$$a\beta\gamma+b\gamma a+c a\beta-(a a+b\beta+c\gamma)\left(\frac{n\cos B.\beta-m\cos C.\gamma}{n-m}\right)=0.$$

It follows that the radical axis of AA', BB', is

$$\frac{n\cos B.\beta-m\cos C.\gamma}{n-m}-\frac{n\cos A.a-l\,\mathrm{los}C.\gamma}{n-l}=0,$$

$$\text{or}\quad (m-n)\cos A.a+(n-l)\cos B.\beta+(l-m)\cos C.\gamma=0,$$

the symmetry of the equation showing that it is the radical axis of all three circles AA', BB', CC'. This radical axis is also the line of the four orthocenters (Art. 3).

8. The radical axis of the diagonal circles is parallel to the Simson Line of D.

If $\lambda a+\mu\beta+\nu\gamma=0$ makes an angle θ with BC, then

$$\cot\theta=\frac{\lambda-\mu\cos C-\nu\cos B}{\nu\sin B-\mu\sin C},$$

so that for the radical axis,

$$a\cot\theta \text{ varies as } \frac{(m-n)}{la-mb\cos C-nc\cos B}.$$

Hence, the radical axis is parallel to the Simson Line (Art. 3).

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

Note on Problems 267 and 268, by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

In the *Messenger of Mathematics,* 1874, Vol. III, page 137, Mr. Glaisher drew attention to the formula:

$$\tan nx=\frac{n\sin x}{\cos x+}\;\frac{(1^2-n^2)\sin^2 x}{3\cos x+}\;\frac{(2^2-n^2)\sin^2 x}{5\cos x+}\;\frac{(3^2-n^2)\sin^2 x}{7\cos x+\ldots} \qquad (1)$$

x being $<\frac{1}{2}\pi$, and n unrestricted.

For $x=\frac{1}{4}\pi$, $\tan\frac{n\pi}{4}=\frac{n}{1+}\,\frac{1^2-n^2}{3+}\,\frac{2^2-n^2}{5+}\,\frac{3^2-n^2}{7+\dots}$

Put $n=\frac{1}{2}$, $\tan\frac{\pi}{8}=\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+\dots}$

Put $n=\frac{1}{3}$, $\tan\frac{\pi}{12}=\frac{3}{9+}\,\frac{6.12}{27+}\,\frac{15.21}{45+}\,\frac{24.30}{63+\dots}$

Put $n=ni$, $\tanh\frac{n\pi}{4}=\frac{n}{1+}\,\frac{n^2+1}{3+\dots}$

and as $e^{2x}=-1+\frac{2}{1-\tanh x}$, this gives a continued fraction for $e^{\frac{1}{2}n\pi}$.

On page 65 of Vol. IV, 1875, Mr. Glaisher states that the value for $\tan nx$ was not, as he had thought, original. Vorsselmann and Heer of Utrecht quoted it in 1833 from Euler (*Mém. de l'Acad. de Pétersbourg*, 1813) in the form

$$\tan nx=\frac{n\tan x}{1-}\,\frac{(n^2-1)\tan^2x}{3-}\,\frac{(n^2-2^2)\tan^2x}{5-}\,\frac{(n^2-3^2)\tan^2x}{7-\dots}. \tag{2}$$

Euler had derived it thus:

$$nz\frac{(1+z)^n+(1-z)^n}{(1+z)^n-(1-z)^n}=1+\frac{(n^2-1)z^2}{3+}\,\frac{(n^2-2^2)z^2}{3+}\,\frac{(n^2-3^2)z^2}{5+\dots}. \tag{3}$$

Vorsellmann derived it from a transformation of

$$\frac{F(\beta+\gamma,\ \beta+1,\ \gamma+1,\ x)}{F(\beta+\gamma,\ \beta,\ \gamma,\ x)}.$$

Glaisher got it from the differential equation corresponding to $y=\cos(n\cos^{-1}x)$ *i. e.*, from $(1-x^2)y_2-xy_1+n^2y=0$ differentiated m times, *i. e.*, $(1-x^2)y_{m+2}-(2m+1)y_{m+1}x+(n^2-m^2)y_m=0$. Then replacing x by $\cos x$, (1) follows at once. Vorsellmann also gave as his own:

$$\tan nx=\frac{nt}{1-t^2-}\,\frac{(n^2-4)t^2}{3(1-t^2)-}\,\frac{(n^2-16)t^2}{5(1-t^2)-\dots} \quad (4), \text{ terminating when } n \text{ is even,}$$

$$=\frac{nt}{1-}\,\frac{(n^2-1)t^2}{3-t^2-}\,\frac{(n^2-9)t^2}{5-3t^2-\dots} \quad (5), \text{ terminating when } n \text{ is odd, } 2t=\tan x.$$

Glaisher points out that (4) is gotten from (2) by substituting $2x$ for x and $\frac{2\tan x}{1-\tan^2 x}$ for $\tan 2x$, and $\frac{1}{2}n$ for n.

In the same way Glaisher gets

$$\tan nx = \frac{nt(1-t^2)}{1-6t^2+t^4-}\,\frac{(n^2-16)t^2\,(1-t^2)^2}{3-18t^2+t^4-}\,\frac{(n^2-64)t^2\,(1-t^2)^2}{5-30t^2+5t^4-\ldots}$$

terminating when n is a multiple of 4.

Putting $n=0$ in (2), (4), (5), we get

$$\tan^{-1}x = \frac{x}{1+}\,\frac{x^2}{3+}\,\frac{4x^2}{5+\ldots}, \quad (6) \quad \text{well known.}$$

$$\tan^{-1}x = \frac{x}{1-x^2+}\,\frac{4x^2}{3(1-x^2)+}\,\frac{16x^2}{5(1-x^2)+\ldots} \quad (7)$$

which is (6) with $\frac{2x}{1-x^2}$ for x.

$$\tan^{-1}x = \frac{x}{1+}\,\frac{x^2}{3-x^2+}\,\frac{9x^2}{5-x^2+\ldots}, \quad (8) \quad \text{due to Euler (1779).}$$

Since $\tan nx$ only changes sign if ni be substituted for n and xi for x, (2) gives us

$$\tan nx = \frac{n\tanh x}{1-}\,\frac{(n^2+1)\tanh^2 x}{3-}\,\frac{(n^2 4)+\tanh^2 x}{5-\ldots}$$

$$\text{or } \tan\left[n \log\sqrt{\frac{1+x}{1-x}}\right] = \frac{nx}{1-}\,\frac{(n^2+1)\,x^2}{3-}\,\frac{(n^2+4)\,x^2}{5-\ldots},$$

a formula also obtainable by replacing n by ni in (3).

Hence, by giving special values to x in (2), (4), and (5), and replacing n by x, we get

$$\tan\frac{\pi x}{4} = \frac{x}{1-}\,\frac{x^2-1}{3-}\,\frac{x^2-4}{5-\ldots} = \frac{x}{1-}\,\frac{x^2-1}{2-}\,\frac{x^2-9}{2-\ldots}$$

$$\tan\frac{\pi x}{3\sqrt{3}} = \frac{x}{1-}\,\frac{x^2-3}{3-}\,\frac{x^2-12}{5-\ldots} = \frac{2x}{3-}\,\frac{4x^2-3}{9-}\,\frac{4x^2-12}{15-\ldots} = \frac{2x}{3-}\,\frac{4x^2-3}{8-}\,\frac{4x^2-27}{12-\ldots}$$

$$\tan\frac{\pi x}{6}=\frac{x}{\sqrt{3}-}\,\frac{x^2-1}{3\sqrt{3}-}\,\frac{x^2-4}{5\sqrt{5}-}$$

For $x=0$, $$\pi=\frac{3\sqrt{3}}{1+}\,\frac{3.1^2}{3+}\,\frac{3.2^2}{5+}\,\frac{3.3^2}{7+\ldots}=\frac{6\sqrt{3}}{3+}\,\frac{3.1^2}{9+}\,\frac{3.2^2}{15+}\,\frac{3.3^2}{21+\ldots}$$

$$=\frac{6\sqrt{3}}{3+}\,\frac{3.1^2}{8+}\,\frac{3.3^2}{12+}\,\frac{3.5^2}{16+\ldots}=\frac{6}{\sqrt{3}+}\,\frac{1}{3\sqrt{3}+}\,\frac{2^2}{5\sqrt{3}+}\,\frac{3^2}{7\sqrt{3}+\ldots.}$$

which may also be deduced from (6), (7), and (8).

Glaisher points out the specific advantage of getting a continued fraction from a differential equation, for not only is the result obtained *ab initio* but the remainder at any stage is exhibited in a finite form. He takes $y=\cos(\sin^{-1}\sqrt{z})$ and its differential equation $4(z-z^2)y_2+2(1-2z)y_1+n^2y=0$. Differentiate m times and we have

$$4(z-z^2)y_{m+2}+2(2m+1)(1-2z)y_{m+1}+(n^2-4m^2)y_m=0.$$

Put $z^2=\sin^2\frac{1}{2}x$ and $2n$ for n and we deduce (4).

From $y=\cos\sqrt{x}$ we get $\tan x=\frac{x}{1-}\,\frac{x^2}{3-\ldots}$

In Vol. VII (1878), page 67, the same author gives further interesting forms:

$$\frac{\log[x+\sqrt{(1+x^2)}]}{\sqrt{(1+x^2)}}=x-\tfrac{2}{3}\,x^3+\frac{2.4}{3.5}x^5-\frac{2.4.6}{3.5.7}x^7+\ldots.$$

$$=\frac{x}{1+}\,\frac{1.2x^2}{3-2x^2+}\,\frac{3.4x^2}{5-4x^2+}\,\frac{5.6x^2}{7-6x^2+\ldots}=\frac{1}{x^{-1}+}\,\frac{1.2}{3x^{-1}-2x+\ldots}$$

$$\frac{\sin^{-1}x}{\sqrt{(1-x^2)}}=x+\tfrac{2}{3}\,x^2+\frac{2.4}{3.5}x^5+\frac{2.4.6}{3.5.7}x^7+\ldots.$$

$$=\frac{x}{1-}\,\frac{1.2x^2}{2x^2+3-}\,\frac{3.4x^2}{4x^2+5-}\,\frac{5.6x^2}{6x^2+7-\ldots}=\frac{1}{x^{-1}-}\,\frac{1.2}{2x+3x^{-1}-\ldots.}$$

Here, putting $x/\sqrt{2}$, we get

$$\frac{2}{\sqrt{3}}\log\frac{1+\sqrt{3}}{\sqrt{2}}=\frac{1}{1+}\,\frac{1}{2+}\,\frac{6}{3+}\,\frac{15}{4+}\,\frac{28}{5+\ldots}$$

1, 6, 15, 28 being the alternate triangular numbers.

$$\frac{\sqrt{2}}{\sqrt{3}}\log\frac{1+\sqrt{3}}{\sqrt{2}}=\frac{1}{\sqrt{2}+}\,\frac{1.2}{2\sqrt{2}+}\,\frac{3.4}{3\sqrt{2}+..}$$

$$\frac{\pi}{2}=\frac{1}{1-}\,\frac{1}{4-}\,\frac{6}{7-}\,\frac{15}{10-}\,\frac{28}{13-...}$$

$$\frac{\pi}{2\sqrt{2}}=\frac{1}{\sqrt{2}-}\,\frac{1.2}{4\sqrt{2}-}\,\frac{3.4}{7\sqrt{2}-}\,\frac{5.6}{10\sqrt{2}-..}$$

For $x=\frac{1}{2}$, $\frac{2}{\sqrt{5}}\log\frac{1+\sqrt{5}}{2}=\frac{1}{2+}\,\frac{1.2}{5+}\,\frac{3.4}{8+}\,\frac{5.6}{11+...}$

$$\frac{\pi}{3\sqrt{3}}=\frac{1}{2-}\,\frac{1.2}{7-}\,\frac{3.4}{12-}\,\frac{5.6}{17-...}$$

For $x=1$ in the expansion of $\frac{\log[x+\sqrt{(1+x^2)}]}{\sqrt{(1+x^2)}}$

$\frac{\log(1+\sqrt{2})}{\sqrt{2}}=\frac{1}{1+}\,\frac{1.2}{1+}\,\frac{3.4}{1+}\,\frac{5.6}{1+...}$, and he compares this with

$$\frac{\pi}{2}-1=\frac{1}{1+}\,\frac{1.2}{1+}\,\frac{2.3}{1+}\,\frac{3.4}{1+}\,\frac{5.6}{1+...}$$

GEOMETRY.

318. Proposed by G. W. GREENWOOD, M. A., Dunbar, Pa.

Is it possible by a straight edge and sect carrier, *i. e.*, without the use of a circle, to construct a mean proportional to two given sects?

Remark by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

The value of the length of the mean proportional can be approximately measured without the application of the circle, but it cannot be constructed by pure geometry without such application.

318. Proposed by G. B. M. ZERR, A. M., Ph. D., 4243 Girard Avenue, Philadelphia, Pa.

Given three radii and the distances apart of the centers of three circles, to find the radii of the eight circles touching the three given circles.

II. Solution by G. W. GREENWOOD, Dunbar, Pa.

Consider first the problem of describing a circle touching two given circles and passing through a given point. Invert with respect to the point; the circles in general invert into circles; draw any common tangent to them